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Status Sum Eigenvalues and Energy of Graphs

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Abstract

Let G be a simple connected graph with vertex set $V(G)$. The status of a vertex $v \in V(G)$ is denoted by $\sigma(v)$ and defined as the sum of distances from v to all other vertices of G. The status sum matrix of G is defined by $S_\sigma(G) = [s_{ij}]$ where $s_{ij} = \sigma(v_i) + \sigma(v_j)$ if $i \neq j$, and $s_{ij} = 0$ otherwise. The status sum energy $E_\sigma(G)$ of a graph G is the sum of the absolute values of the eigenvalues of the status sum matrix. In this paper, we obtain some coefficients of the characteristic polynomial $\Phi(G, \mu)$ of status sum matrix. Status sum energies of some well known graphs are obtained and some upper and lower bounds for $E_\sigma(G)$ are established.

AMS subject classification: 05C50.

Keywords: Distance in graphs, status of a vertex, status sum matrix, status sum eigenvalues, status sum energy of a graph.

1 Introduction

In this paper, all the graphs are assumed to be finite, connected and simple. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We call $|V| = n$ to be the order and $|E| = m$ to be the size of a graph G. The distance $d(u, v)$ between any two vertices u and v in a graph G is the length of a shortest path connecting them. The status of a vertex $v \in V(G)$ is denoted by $\sigma(v)$ and defined as the sum of all distances from v to

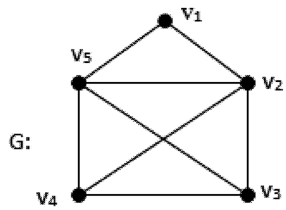

Figure 1: Graph G

all other vertices of G [15]. That is

$$\sigma(v) = \sum_{u \in V(G)} d(u, v). \tag{1}$$

A graph G is called r-status distance balanced regular graph or status transmission regular graph if $\sigma(v) = r$ for each $v \in V(G)$.

Let $v_1, v_2, \ldots, v_n$ be the vertices of G. The status sum matrix of G is a square matrix $S_\sigma(G) = [s_{ij}]$ of order n, where

$$s_{ij} = \begin{cases} \sigma(v_i) + \sigma(v_j), & \text{if } i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $S_\sigma(G)$ is denoted by $\Phi(G, \mu)$. The eigenvalues of $S_\sigma(G)$ are denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The status sum energy of a graph G is defined as the sum of the absolute values of the eigenvalues of $S_\sigma(G)$. That is

$$E_\sigma(G) = \sum_{i=1}^{n} |\mu_i|. \tag{2}$$

Equation (2) is analogous to the ordinary graph energy defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G [7]. There are many results and applications on different types of graph energy, see [6] for a study on degree based energies of graphs.

Example 1.1. *The status sum matrix of a graph given in Fig. 1 is*

$$S_\sigma(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 10 & 11 & 11 & 10 \\ 10 & 0 & 9 & 9 & 8 \\ 11 & 9 & 0 & 10 & 9 \\ 11 & 9 & 10 & 0 & 9 \\ 10 & 8 & 9 & 9 & 0 \end{pmatrix} \end{array}$$

The eigenvalues of status sum matrix are -11.5287, -10.0000, -8.9750, -8.0000 *and* 38.5037. *Therefore* $E_\sigma(G) = 77.0074$.

Several graph energies were introduced and studied in the literature such as distance energy [10], Laplacian energy [8], degree sum energy [14].

In this paper we study the characteristic polynomial and eigenvalues of the status sum matrix of a graph. Also the bounds for the status sum energy of a graph are obtained.

2 Status sum eigenvalues

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $S_\sigma(G)$. Since trace of $S_\sigma(G) = 0$, the eigenvalues of $S_\sigma(G)$ satisfy the relations

$$\sum_{i=1}^{n} \mu_i = 0 \tag{3}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \mu_i^2 &= \text{trace}[(S_\sigma(G))^2] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}s_{ji} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} (s_{ij})^2 = 2\sum_{i<j} (\sigma(v_i) + \sigma(v_j))^2 = 2M. \end{aligned} \tag{4}$$

Theorem 2.1. *If* G *is an* r*-status distance balanced regular graph with* n*, then* $S_\sigma(G)$ *has only one positive eigenvalue equal to* $2(n-1)r$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Since G is an r-status distance balanced regular graph, $\sigma(v_i) = r$, for $i = 1, 2, \ldots, n$. Therefore $s_{ij} = \sigma(v_i) + \sigma(v_j) = 2r$ if $i \neq j$ and $s_{ij} = 0$, otherwise. Therefore the characteristic polynomial of the status sum matrix is

$$\begin{aligned}\Phi(G:\mu) &= \det(\mu I - S_\sigma(G)) \\ &= \det\left(\mu I - 2r(A(K_n))\right), \\ &\quad \text{where } A(K_n) \text{ is the adjacency matrix of a complete graph } K_n \\ &= (2r)^n \left|\frac{\mu}{2r} I - A(K_n)\right| \\ &= (2r)^n \left(\frac{\mu}{2r} - n + 1\right)\left(\frac{\mu}{2r} + 1\right)^{n-1} \\ &= [\mu - 2(n-1)r]\,(\mu + 2r)^{n-1}.\end{aligned}$$

Hence the result follows. □

Theorem 2.2. *If G is a connected graph with n vertices, then*

$$\mu_1 \leq \sqrt{\frac{2M_1(n-1)}{n}}. \tag{5}$$

Proof. The Cauchy-Schwarz inequality states that if $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two real n-vectors, then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Let $a_i = 1$ and $b_i = \mu_i$ for $i = 2, 3, \ldots, n$. Therefore

$$\left(\sum_{i=2}^{n} \mu_i\right)^2 \leq (n-1)\left(\sum_{i=2}^{n} \mu_i^2\right). \tag{6}$$

From Eqn. (3) and Eqn. (4), we have $\sum_{i=2}^{n} \mu_i = -\mu_1$ and $\sum_{i=2}^{n} \mu_i^2 = 2M_1 - \mu_1^2$. Therefore Eq. (6) becomes

$$(-\mu_1)^2 \leq (n-1)\left(2M_1 - \mu_1^2\right),$$

which gives

$$\mu_1 \leq \sqrt{\frac{2M_1(n-1)}{n}}.$$

□

Equality in Eqn. (5) holds for status distance balanced regular graphs.

3 Characteristic polynomial of status sum matrix

In this section, we obtain certain coefficients of the characteristic polynomial of the status sum matrix of a graph.

Theorem 3.1. *Let G be a graph of order n and let*

$$\Phi(G,\mu) = c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \cdots + c_n$$

be the characteristic polynomial of $S_\sigma(G)$. Then

$$c_0 = 1 \quad \text{and} \quad c_1 = 0.$$

$$\begin{aligned}
c_2 &= -\sum_{i=1,i\neq j}^{n} \left(\sigma(v_i)+\sigma(v_j)\right)^2 . \\
c_3 &= -2\sum_{i=1,i\neq j}^{n} \big[2\sigma(v_i)\sigma(v_j)(\sigma(v_k) + \sigma(v_i)^2(\sigma(v_j)+\sigma(v_k)) + \sigma(v_j)^2(\sigma(v_i)+\sigma(v_k)) \\
&\quad +\sigma(v_k)^2(\sigma(v_i)+\sigma(v_j))\big] . \\
c_4 &= \sum_{i=1,i\neq j}^{n} -4\left[\sigma(v_i)^2(\sigma(v_j)\sigma(v_l)+\sigma(v_j)\sigma(v_k)+\sigma(v_l)\sigma(v_k))\right. \\
&\quad +\sigma(v_k)^2(\sigma(v_i)\sigma(v_l)+\sigma(v_j)\sigma(v_l)+\sigma(v_i)\sigma(v_j)) \\
&\quad +\sigma(v_l)^2(\sigma(v_i)\sigma(v_k)+\sigma(v_i)\sigma(v_j)+\sigma(v_k)\sigma(v_j)) \\
&\quad \left.\sigma(v_j)^2(\sigma(v_k)\sigma(v_l)+\sigma(v_i)\sigma(v_k)+\sigma(v_i)\sigma(v_l))\right] . \\
c_n &= (-1)^n det(S_\sigma(G)).
\end{aligned}$$

Proof. The proof of c_0 and c_1 are similar to the proof in [1].

$$\begin{aligned}
c_2 &= (-1)^2 \sum_{1\leq i<j\leq n} \begin{vmatrix} 0 & s_{ij} \\ s_{ji} & 0 \end{vmatrix} \\
&= -\sum_{1\leq i<j\leq n} s_{ij}^2 \\
&= -\sum_{1\leq i<j\leq n} \left(\sigma(v_i)+\sigma(v_j)\right)^2 .
\end{aligned}$$

We then have

$$\begin{aligned}
c_3 &= (-1)^3 \sum_{1\leq i<j<k\leq n} \begin{vmatrix} 0 & s_{ij} & s_{ik} \\ s_{ji} & 0 & s_{jk} \\ s_{ki} & s_{kj} & 0 \end{vmatrix} \\
&= -2\sum_{1\leq i<j<k\leq n} (s_{ij}s_{jk}s_{ik}) \\
&= -2\sum_{1\leq i<j<k\leq n} [(\sigma(v_i)+\sigma(v_j))(\sigma(v_j)+\sigma(v_k))(\sigma(v_k)+\sigma(v_i))]
\end{aligned}$$

We similarly have

$$
\begin{aligned}
c_4 &= (-1)^4 \sum_{1\le i<j<k<l\le n} \begin{vmatrix} s_{ii} & s_{ij} & s_{ik} & s_{il} \\ s_{ji} & s_{jj} & s_{jk} & s_{jl} \\ s_{ki} & s_{kj} & s_{kk} & s_{kl} \\ s_{li} & s_{lj} & s_{lk} & s_{ll} \end{vmatrix} \\
&= \sum_{1\le i<j<k<l\le n} \left[s_{ii} \begin{vmatrix} s_{jj} & s_{jk} & s_{jl} \\ s_{kj} & s_{kk} & s_{kl} \\ s_{lj} & s_{lk} & s_{ll} \end{vmatrix} - s_{ij} \begin{vmatrix} s_{ii} & s_{ik} & s_{il} \\ s_{ki} & s_{kk} & s_{kl} \\ s_{li} & s_{lk} & s_{ll} \end{vmatrix} + s_{ik} \begin{vmatrix} s_{ii} & s_{ij} & s_{il} \\ s_{ji} & s_{jj} & s_{jl} \\ s_{li} & s_{lj} & s_{ll} \end{vmatrix} \right. \\
&\quad \left. -s_{il} \begin{vmatrix} s_{ii} & s_{ij} & s_{ik} \\ s_{ji} & s_{jj} & s_{jk} \\ s_{ki} & s_{kj} & s_{kk} \end{vmatrix} \right] \\
&= \sum_{1\le i<j<k<l\le n} -4 \left[\sigma(v_i)^2(\sigma(v_j)\sigma(v_l) + \sigma(v_j)\sigma(v_k) + \sigma(v_l)\sigma(v_k)) \right. \\
&\quad +\sigma(v_k)^2(\sigma(v_i)\sigma(v_l) + \sigma(v_j)\sigma(v_l) + \sigma(v_i)\sigma(v_j)) \\
&\quad +\sigma(v_l)^2(\sigma(v_i)\sigma(v_k) + \sigma(v_i)\sigma(v_j) + \sigma(v_k)\sigma(v_j)) \\
&\quad \left. \sigma(v_j)^2(\sigma(v_k)\sigma(v_l) + \sigma(v_i)\sigma(v_k) + \sigma(v_i)\sigma(v_l)) \right].
\end{aligned}
$$

Finally

$$c_n = (-1)^n \det(S_\sigma(G)).$$

□

Remark 3.2. *1. The number of terms in c_3 in the above theorem is equal to number of triangles in the graph.*

2. If $g(G) \neq 3$, then $c_3 = 0$.

Theorem 3.3. *If $\mu_1, \mu_2, \ldots, \mu_n$ are the status sum eigenvalues of a graph G, then*

$$\sum_{i=1}^{n} \mu_i^2 = -2c_2. \tag{7}$$

Proof. We have

$$\begin{aligned}\sum_{i=1}^{n}\mu_i^2 &= \text{trace}(S_\sigma^2(G))\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} s_{ik}s_{ki}\\ &= 2\sum_{i=1}^{n}\sum_{k=1}^{n} s_{ik}^2\\ &= 2\sum_{1\le i<k\le n} s_{ik}^2\\ &= 2\sum_{1\le i<k\le n} (\sigma(v_i)+\sigma(v_k))^2 .\end{aligned}$$

Hence

$$\sum_{i=1}^{n}\mu_i^2 = -2c_2.$$

□

Theorem 3.4. *If G is a complete graph of order n, then*

$$c_2 = -2n(n-1)^3. \tag{8}$$

Proof. For any vertex u in the complete graph of order n, $\sigma(u) = n-1$. Therefore by Theorem 3.1

$$\begin{aligned}c_2 &= -\sum_{1\le i<j\le n}(2n-2)^2\\ &= -2n(n-1)^3.\end{aligned}$$

□

Theorem 3.5. *For a complete bipartite graph $K_{p,q}$, we have*

$$c_2 = -\left[pq(3(p+q)-4)^2 + 2p(p-1)(q+2(p-1))^2 + 2q(q-1)(p+2(q-1))^2\right]. \tag{9}$$

Proof. The graph $K_{p,q}$ has $n = p+q$ vertices and $m = pq$ edges. Also $\text{diam}(K_{p,q}) \le 2$. The vertex set $V(K_{p,q})$ can be partitioned into two sets V_1 and V_2 such that for every edge uv of $K_{p,q}$, the vertex $u \in V_1$ and $v \in V_2$, where $|V_1| = p$ and $|V_2| = q$. Therefore as $d(u) = q$ and $d(v) = p$, we have

$$\begin{aligned}\sigma(u) &= q+2(p-1),\ \ u\in V_1,\\ \sigma(v) &= p+2(q-1),\ \ u\in V_2.\end{aligned}$$

We then obtain

$$
\begin{aligned}
c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\
&= -\left[\sum_{v_iv_j\in E(G)} (\sigma(v_i)+\sigma(v_j))^2 + \sum_{v_i,v_j\in V_1; i\neq j} (\sigma(v_i)+\sigma(v_j))^2\right. \\
&\quad \left. + \sum_{v_i,v_j\in V_2; i\neq j} (\sigma(v_i)+\sigma(v_j))^2]\right] \\
&= -\left[m\,(q+2(p-1)+p+2(q-1))^2 + \binom{p}{2}2(q+2(p-1))^2 + \binom{q}{2}2(p+2(q-1))^2\right] \\
&= -\left[pq\,((p+q)+2(p+q-2))^2 + \frac{p(p-1)}{2}\left(2q+4(p-1)^2\right) + \frac{q(q-1)}{2}4(p+2(q-1))^2\right] \\
&= -\left[pq\,(3(p+q)-4)^2 + p(p-1)2(q+2(p-1))^2 + q(q-1)2(p+2(q-1))^2\right] \\
&= -\left[pq\,(3(p+q)-4)^2 + 2p(p-1)(q+2(p-1))^2 + 2q(q-1)(p+2(q-1))^2\right].
\end{aligned}
$$

□

Corollary 3.6. *For the star graph $K_{1,r}$, $r\ge 2$, we have*

$$c_2 = -\left[r(3(1+r)-4)^2 + 2r(r-1)(1+2(r-1))^2\right].$$

Theorem 3.7. *For a path P_n on n vertices*

$$c_2 = -\sum_{1\le i<j\le n}\left[(n^2+n)+i(i-n-1)+j(j-n-1)\right]^2.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n where v_i is adjacent to v_{i+1}, $i = 1, 2, \ldots, n-1$. Therefore for $i = 1, 2, \ldots, n-1$, we have

$$\sigma(v_i) = \frac{n^2+n}{2} + i(i-n-1).$$

Therefore

$$
\begin{aligned}
c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\
&= -\sum_{1\le i<j\le n}\left[(n^2+n)+i(i-n-1)+j(j-n-1)\right]^2.
\end{aligned}
$$

□

Theorem 3.8. *For a wheel graph W_{n+1}, $n\ge 3$, we have*

$$c_2 = -[9n(n-1)^2 + 2n(n-1)(2n-3)^2].$$

Proof. The wheel graph W_{n+1} has $n+1$ vertices and $2n$ edges. Also $\text{diam}(W_{n+1}) = 2$. The degree of a central vertex is n and all other vertices has degree 3. Therefore if u is a central vertex of W_{n+1}, then $\sigma(u) = n$ and if v is avertex of degree 3, then $\sigma(v) = 3+2(n+1-4) = 2n-3$. Hence

$$\begin{aligned}
c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\
&= -\left[\sum_{d(u)=n;d(v)=3} (\sigma(u)+\sigma(v))^2 + \sum_{d(u)=3;d(v)=3} (\sigma(u)+\sigma(v))^2\right] \\
&= -\left[\sum_{d(u)=n;d(v)=3} (n+2n-3)^2 + \sum_{d(u)=3;d(v)=3} (2(2n-3))^2\right] \\
&= -\left[n(3n-3)^2 + \binom{n}{2} 4(2n-3)^2\right] \\
&= -[9n(n-1)^2 + 2n(n-1)(2n-3)^2].
\end{aligned}$$

□

A *friendship graph* (or *Dutch windmill graph*) F_n, $n \ge 2$, is a graph that can be constructed by coalescence n copies of the cycle C_3 of length 3 with a common vertex. It has $2n+1$ vertices and $3n$ edges. The degree of a coalescence vertex of F_n is $2n$ and the degree of all other vertices is 2.

Theorem 3.9. *For the friendship graph F_n with $n \ge 2$, we have*

$$c_2 = -[8n(3n-1)^2 + 4n(2n-1)(2n-1)^2].$$

Proof. The friendship graph F_n has $2n+1$ vertices and $3n$ edges. Also $\text{diam}(F_n) = 2$. The degree of a coalescence vertex is $2n$ and all other vertices has degree 2. Therefore if u is a coalescence vertex of F_n, then $\sigma(u) = 2n$ and if v is avertex of degree 2, then $\sigma(v) = 2+2(2n-2) = 4n-2$. Hence

$$\begin{aligned}
c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\
&= -\left[\sum_{d(u)=2n;d(v)=2} (\sigma(u)+\sigma(v))^2 + \sum_{d(u)=2;d(v)=2} (\sigma(u)+\sigma(v))^2\right] \\
&= -\left[\sum_{d(u)=2n;d(v)=2} (2n+4n-2)^2 + \sum_{d(u)=2;d(v)=2} (2(4n-2))^2\right] \\
&= -\left[2n(6n-2)^2 + \binom{2n}{2} 4(2n-1)^2\right] \\
&= -[8n(3n-1)^2 + 4n(2n-1)(2n-1)^2].
\end{aligned}$$

□

Theorem 3.10. *Let G be a graph of order n and size m. Then*

$$(n-1)^2 \leq \frac{-c_2}{4\binom{n}{2}} \leq \binom{n}{2}^2.$$

Proof. For any vertex $u_i \in V(G)$, we have

$$(n-1) \leq \sigma(u_i) \leq \binom{n}{2}.$$

Hence we can write

$$(n-1)+(n-1) \leq \sigma(u_i)+\sigma(u_j) \leq 2\binom{n}{2}, \quad i \neq j$$

and hence we find

$$2(n-1) \leq \sigma(u_i)+\sigma(u_j) \leq 2\binom{n}{2}.$$

Taking squares, we get

$$\begin{aligned}
4(n-1)^2 \leq (\sigma(u_i)+\sigma(u_j))^2 &\leq 4\binom{n}{2}^2 \\
4\sum_{1\leq i<j\leq n}(n-1)^2 \leq \sum_{1\leq i<j\leq n}(\sigma(u_i)+\sigma(u_j))^2 &\leq 4\sum_{1\leq i<j\leq n}\binom{n}{2}^2 \\
4(n-1)^2\sum_{1\leq i<j\leq n}1 \leq -c_2 &\leq 4\binom{n}{2}^2\sum_{1\leq i<j\leq n}1 \\
4(n-1)^2\binom{n}{2} \leq -c_2 &\leq 4\binom{n}{2}^2\binom{n}{2} \\
4(n-1)^2\binom{n}{2} \leq -c_2 &\leq 4\binom{n}{2}^3 \\
(n-1)^2 \leq \frac{-c_2}{4\binom{n}{2}} &\leq \binom{n}{2}^2.
\end{aligned}$$

□

Theorem 3.11. *Let G be a k-status distance balanced regular graph with n vertices and m edges. Then*

$$c_2 = 2n(1-n)k^2.$$

Proof. As we know that

$$c_2 = -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2,$$

since G is k-status distance balanced regular graph, we have $\sigma(v_i) = k$, for each $i = 1,2,\dots,n$. Hence,

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le n} (k+k)^2 \\ &= -\binom{n}{2}(2k)^2 \\ &= 2n(1-n)k^2. \end{aligned}$$

□

Corollary 3.12. *For a cycle C_n, $n \ge 3$, we have*

$$c_2 = \begin{cases} \frac{n^5(1-n)}{8}, & if \;\; n \;\; is \;\; even, \\ \frac{n(1-n)(n^2-1)^2}{8}, & if \;\; n \;\; is \;\; odd. \end{cases} \tag{10}$$

Proof. We have

$$c_2 = -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2,$$

and

$$\sigma(v_i) = \begin{cases} \frac{n^2}{4}, & if \;\; n \;\; is \;\; even, \\ \frac{n^2-1}{4}, & if \;\; n \;\; is \;\; odd. \end{cases}$$

If n is even, then

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le n}^{n} \left(\frac{n^2}{4}+\frac{n^2}{4}\right)^2 \\ &= -\binom{n}{2}\frac{n^4}{4} \\ &= \frac{n^5(1-n)}{8}. \end{aligned}$$

If n is odd, then

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\ &= -\binom{n}{2}\left(\frac{n^2-1}{4}+\frac{n^2-1}{4}\right)^2 \\ &= \frac{n(1-n)(n^2-1)^2}{8}. \end{aligned}$$

□

Let S be a set and $F = \{S_1, S_2, S_3, \cdots, S_q\}$ be a non-empty family of distinct non-empty subset of S such that $S = \bigcup_{i=1}^{q} S_i$. The intersection graph graph of S which is denoted by $\Omega(F)$ has F as its set of vertices and two distinct vertices $S_i, S_j, i \neq j$, are adjacent if and only if $S_i \cap S_j \neq \phi$. Here we will consider a set S of cardinality p and let F be the set of all subsets of S of cardinality t, $1 < t < p$, which is denoted by S^t. Upon convinience we may set $S = \{1, 2, \ldots, p\}$. Let us denote the intersection graph $\Omega(S^t)$ by $\Gamma^t = (V, E)$. The number of vertices of this graph is $|V| = \binom{p}{t}$

Corollary 3.13. *Let $S = \{1, 2, 3, \cdots, p\}$ be any set and let $F = \{S_1, S_2, \cdots, S_t\}$ be the family of subsets of S such that $S = S_1 \cup S_2 \cup \cdots \cup S_t$. Let $\Omega(S^t) = \Gamma^t$ be the intersection graph of S. Then*

$$c_2 = \begin{cases} -\binom{\binom{p}{t}}{2}\left[4\left(\binom{p}{t} + \binom{p-t}{t} - 1\right)^2\right] & for \;\; p \geq 2t, \\ -\binom{\binom{p}{t}}{2}\left[4\left(\binom{p}{t} - 1\right)^2\right] & for \;\; p < 2t. \end{cases} \tag{11}$$

Proof. We have,

$$c_2 = -\sum_{1 \leq i < j \leq n} (\sigma(v_i) + \sigma(v_j))^2 .$$

By [5], the intersection graph $\Gamma^{\{t\}}$ is vertex-transitive and for any t-element subset A of S we have

$$\sigma_{\Gamma^{\{t\}}}(A) = \begin{cases} \binom{p}{t} + \binom{p-t}{t} - 1, & p \geq 2t \\ \binom{p}{t} - 1, & p < 2t. \end{cases}$$

For $p \geq 2t$,

$$\begin{aligned} c_2 &= -\sum_{1 \leq i < j \leq n} (\sigma(v_i) + \sigma(v_j))^2 \\ &= -\left[\sum_{1 \leq i < j \leq n} \left(2\left(\binom{p}{t} + \binom{p-t}{t} - 1\right)\right)^2\right] \\ &= -\binom{\binom{p}{t}}{2}\left[4\left(\binom{p}{t} + \binom{p-t}{t} - 1\right)^2\right]. \end{aligned}$$

For $p < 2t$,

$$\begin{aligned} c_2 &= -\sum_{1 \leq i < j \leq n} (\sigma(v_i) + \sigma(v_j))^2 \\ &= -\sum_{1 \leq i < j \leq n} \left(2\left(\binom{p}{t} - 1\right)\right)^2 \\ &= -\binom{\binom{p}{t}}{2}\left[4\left(\binom{p}{t} - 1\right)^2\right]. \end{aligned}$$

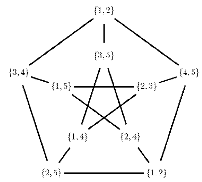

Figure 2. The odd graph $O_3 = KG_{5,2}$ is isomorphic to the Petersen graph

□

The vertex set of the hypercube H_n consists of all n-tuples $(b_1, b_2, \ldots, b_n)$ with $b_i \in \{0,1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one place. Moreover, H_n has exactly $2n$ vertices and $n2^{n-1}$ edges. Darafsheh, [5], proved that H_n is vertex transitive and for every vertex u, $\sigma_{H_n}(u) = n2^{n-1}$.

Corollary 3.14. *For a hypercube H_n,*

$$c_2 = -4n^3(2n-1)2^{2(n-1)}. \tag{12}$$

Proof. From [5], $|V(H_n)| = 2n$ and we have

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le 2n} (\sigma(v_i)+\sigma(v_j))^2 \\ &= -\sum_{1\le i<j\le 2n} \left(2(n2^{n-1})\right)^2 \\ &= -\binom{2n}{2} 4n^2 2^{2(n-1)} \\ &= -4n^3(2n-1)2^{2(n-1)}. \end{aligned}$$

□

The Kneser graph $KG_{p,k}$ is the graph whose vertices correspond to the k-element subsets of a set of p elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. Clearly we must impose the restriction $p \ge 2k$. The Kneser graph $KG_{p,k}$ has $\binom{p}{k}$ vertices and it is regular of degree $\binom{p-k}{k}$. Therefore the number of edges of $KG_{p,k}$ is $\frac{1}{2}\binom{p}{k}\binom{p-k}{k}$ (see [13]). The kneser graph $KG_{n,1}$ is a complete graph on n vertices. The Kneser graph $KG_{n,1}$ is the complete graph on n vertices. The Kneser graph $KG_{2p-1,p-1}$ is known as the odd graph O_p. The odd graph $O_3 = KG_{5,2}$ is isomorphic to the Petersen graph (see Fig. 2).

Corollary 3.15. *For the Kneser graph $KG_{p,k}$, we have*

$$c_2 = -16\binom{\binom{p}{k}}{2}\left(\frac{W(KG_{p,k})}{\binom{p}{k}}\right)^2. \tag{13}$$

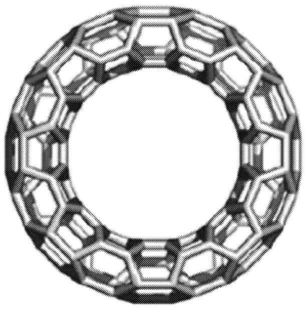

Figure 3. An achiral polyhex nanotorus (or toroidal fullerene) $T[p,q]$.

Proof. From [13], we have

$$\sigma_{KG_{p,k}}(A) = \frac{2W(KG_{p,k})}{\binom{p}{k}}.$$

We then have

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\ &= -\sum_{1\le i<j\le n} \left(2\left(\frac{2W(KG_{p,k})}{\binom{p}{k}}\right)\right)^2 \\ &= -4\binom{\binom{p}{k}}{2} 4\left(\frac{2W(KG_{p,k})}{\binom{p}{k}}\right)^2 \\ &= -16\binom{\binom{p}{k}}{2}\left(\frac{2W(KG_{p,k})}{\binom{p}{k}}\right)^2. \end{aligned}$$

□

A nanostructure called an achiral polyhex nanotorus (or toroidal fullerene) of perimeter p and length q, denoted by $T[p,q]$ is depicted in Fig. 3 and its 2-dimensional molecular graph is in Fig. 4. It is regular of degree 3 and has pq vertices and $\frac{3pq}{2}$ edges.

Corollary 3.16. *For an achiral polyhex nanotorus $T = T[p,q]$, we have*

$$c_2 = \begin{cases} -\binom{pq}{2}\left(\frac{q}{6}(6p^2+q^2-4)\right)^2 & for \;\; q<p, \\ -\binom{pq}{2}\left(\frac{p}{6}(3q^2+3pq+p^2-4)\right)^2 & for \;\; q\ge p. \end{cases}$$

Proof. From [2, 16], for any vertex $u \in V(T)$, we have

$$\sigma_T(u) = \begin{cases} \frac{q}{12}(6p^2+q^2-4) & for \;\; q<p, \\ \frac{p}{12}(3q^2+3pq+p^2-4) & for \;\; q\ge p. \end{cases}$$

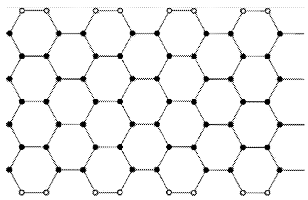

Figure 4. A 2-dimensional lattice for an achiral polyhex nanotorus $T[p,q]$.

We have

$$c_2 = -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 .$$

For $q < p$,

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\ &= -\sum_{1\le i<j\le n} \left(2\left(\frac{q}{12}(6p^2+q^2-4)\right)\right)^2 \\ &= -\binom{pq}{2}\left(\frac{q}{6}(6p^2+q^2-4)\right)^2 . \end{aligned}$$

For $q \ge p$,

$$\begin{aligned} c_2 &= -\sum_{1\le i<j\le n} (\sigma(v_i)+\sigma(v_j))^2 \\ &= -\sum_{1\le i<j\le n} \left(2\left(\frac{p}{12}(3q^2+3pq+p^2-4)\right)\right)^2 \\ &= -\binom{pq}{2}\left(\frac{p}{6}(3q^2+3pq+p^2-4)\right)^2 . \end{aligned}$$

□

4 Bounds for the status sum energy of graphs

Theorem 4.1. *If G is an r-status distance balanced regular graph with n vertices, then the status sum energy of a graph G is equal to $4(n-1)r$.*

Proof. Let G be a r- status distance balanced regular graph with n vertices. From Theorem 2.1, G has only one positive eigenvalue which equals to $2(n-1)r$. Hence $E_\sigma(G) = 4(n-1)r$. □

Theorem 4.2. *If G is any graph with n vertices, then*

$$\sqrt{2M} \leq E_\sigma(G) \leq \sqrt{2Mn}. \tag{14}$$

Proof. Put $a_i = 1$ and $b_i = \mu_i$ for $i = 1, 2, \ldots, n$ in Cauchy-Schwarz inequality. Therefore

$$\left(\sum_{i=1}^{n} \mu_i\right)^2 \leq n\left(\sum_{i=1}^{n} \mu_i^2\right).$$

From which,

$$[E_\sigma(G)]^2 \leq n(2M)$$

$$E_\sigma(G) \leq \sqrt{2nM}. \tag{15}$$

This is an upper bound. Now

$$\begin{aligned} [E_\sigma(G)]^2 &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \\ &\geq \sum_{i=1}^{n} |\mu_i|^2 = 2M, \end{aligned}$$

which gives a lower bound

$$E_\sigma(G) \geq \sqrt{2M}. \tag{16}$$

Combining Eq.(15) and Eq. (16)we get

$$\sqrt{2M} \leq E_\sigma(G) \leq \sqrt{2Mn}.$$

□

Theorem 4.3. *Let G be any graph with n vertices and let $\triangle$ be the absolute value of the determinant of the status sum matrix $S_\sigma(G)$. Then*

$$\sqrt{2M + n(n-1)\triangle^{2/n}} \leq E_\sigma(G)\sqrt{2Mn}. \tag{17}$$

Proof. For the lower bound. By the definition of status sum energy

$$\begin{aligned} [E_\sigma(G)]^2 &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \\ &= \sum_{i=1}^{n} (\mu_i)^2 + 2\sum_{i<j} |\mu_i||\mu_j|. \end{aligned}$$

By Eqn. (4), we have

$$\begin{aligned} [E_\sigma(G)]^2 &= 2M + 2\sum_{i<j} |\mu_i||\mu_j| \\ &= 2M + \sum_{i\neq j} |\mu_i||\mu_j|. \end{aligned} \tag{18}$$

Since for non-negative numbers, the arithmetic mean is not smaller than the geometric mean, we get

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{i\neq j} |\mu_i||\mu_j| &\geq \left(\prod_{i\neq j} |\mu_i||\mu_j| \right)^{1/n(n-1)} \\ &= \left(\prod_{i=1}^{n} |\mu_i|^{2(n-1)} \right)^{1/n(n-1)} \\ &= \prod_{i=1}^{n} |\mu_i|^{2/n} \\ &= \triangle^{2/n}. \end{aligned}$$

Therefore,

$$\sum_{i\neq j} |\mu_i||\mu_j| \geq n(n-1)\triangle^{2/n}. \tag{19}$$

Combining Eq. (18) and Eq. (19), we get,

$$[E_\sigma(G)]^2 \geq 2M + n(n-1)\triangle^{2/n}$$

$$E_\sigma(G) \geq \sqrt{2M + n(n-1)\triangle^{2/n}}.$$

For the upper bound. Consider the quantity X whose value is evidently non-negative

$$\begin{aligned} X &= \sum_{i=1}^{n}\sum_{j=1}^{m} (|\mu_i| - |\mu_j|)^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \left[|\mu_i|^2 + |\mu_j|^2 - 2|\mu_i||\mu_j| \right]. \end{aligned}$$

By direct expansion,

$$X = n\sum_{i=1}^{n}(\mu_i)^2 + n\sum_{j=1}^{n}(\mu_j)^2 - 2\left(\sum_{i=1}^{n}|\mu_i|\right)\left(\sum_{j=1}^{n}|\mu_j|\right),$$

which in view of Eq. (2) and Eq. (4), the above equation yields

$$\begin{aligned} X &= 2nM + 2nM - 2\left[E_\sigma(G)\right]^2 \\ &= 4nM - 2\left[E_\sigma(G)\right]^2. \end{aligned}$$

Since $X \geq 0$,

$$\begin{aligned} 4nM - 2\left[E_\sigma(G)\right]^2 &\geq 0 \\ \left[E_\sigma(G)\right]^2 \leq 2nM \\ E_\sigma(G) \leq \sqrt{2nM}. \end{aligned}$$

□

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